SOME THEOREMS CONCERNING THE NATURAL FREQUENCIES OF LINEAR SYSTEMS

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CHAPTER I

INTRODUCTION

On pages 32, 33, and 286 of Volume I of Courant-Hilbert's Methods of Mathematical Physics [1], there is a set of theorems concerning the eigenvalues of quadratic forms and the natural frequencies of dissipationless linear systems of n degrees of freedom executing small free vibrations about a position of stable equilibrium. One of these theorems (page 33) is so imprecisely worded that its meaning is difficult to decipher, and the proofs of four others (page 286) are so sketchy that one can almost claim that the proofs have been omitted. The object of the present paper is to clarify the imprecisely worded theorem by restating it carefully in terminology currently in use, to prove all of the theorems, and to present numerical examples illustrating the application of several of them to some simple mechanical systems.

In Chapter II three essentially mathematical theorems and a corollary are proved. Theorems 1 and 2 are modifications of well-known proofs [2]; they characterize the eigenvalues of a linear transformation $P^{-1}T$, where T is symmetric and P is positive definite, by using a minimum principle and a maximum-minimum principle, respectively. Theorem 3 is a restatement of the allegedly ambiguous theorem on page 33 of Courant-Hilbert concerning the relations between the eigenvalues of a quadratic form and those of a second quadratic form obtained from the first by imposing constraints on the unknowns. The corollary mentioned concerns the effect of constraints of a particularly simple form.

In Chapter III Theorems 1-3 are used to prove four theorems concerning the natural frequencies of linear systems. The first two of these theorems relate the natural frequencies of a system to the natural frequencies of a second system obtained from the first by imposing constraints on the unknowns. Theorem 6 establishes a relation between the natural frequencies of two systems in which the potential energies are the same, but the kinetic energy of one is greater than or equal to the kinetic energy of the other, whereas Theorem 7 relates the natural frequencies of two systems in which the kinetic energies are the same, but the potential energy of one is greater than or equal to the potential energy of the other.

Chapter IV consists of four examples that illustrate Theorems 5-7 either by exhibiting the natural frequencies of certain systems or by proving analytically that the natural frequencies have certain properties. Example 1 illustrates the effect of stiffening a system (Theorem 7); Examples 2 and 4 illustrate the effect of increasing the inertia (Theorem 6); and Examples 3 and 4 illustrate the effect of imposing a constraint (Theorem 5).

CHAPTER II

BASIC MATHEMATICAL THEOREMS

In this chapter three mathematical theorems and a corollary are proved. The first two theorems, which are modifications of well-known results [2], describe different methods for determining the eigenvalues of the transformation $P^{-1}T$, where P is a symmetric positive definite transformation and T is symmetric. Theorem 3 is a restatement of the supposedly ambiguous theorem on page 33 of Courant-Hilbert. This theorem relates the eigenvalues of a quadratic form to the eigenvalues of a second quadratic form obtained from the first by imposing constraints on the unknowns. The corollary mentioned is an immediate consequence of Theorem 3 when the constraints are of a simple nature.

<u>Definition 1</u>: Let E_n be a real n-dimensional Euclidean vector space with orthonormal basis $\delta_1,\ldots,\delta_n$, where the k^{th} coordinate of δ_k $(k = 1,\ldots,n)$ is 1 and the other coordinates are zero. Let T and P be symmetric transformations on E_n to E_n with domain E_n, and let P be positive definite. Let the matrix representations of T and P in the δ-basis be

$$T \overset{\delta}{\longleftrightarrow} [t_{ij}] \quad \text{and} \quad P \overset{\delta}{\longleftrightarrow} [p_{ij}] .$$

The real numbers s_k $(k = 1,\ldots,n)$ such that $T\alpha_k = s_k P\alpha_k$ $(k = 1,\ldots,n)$ are called <u>the generalized eigenvalues of T with respect to P</u>, and the vectors α_k are <u>the generalized eigenvectors of T with respect to P</u>.

Note: The s_k are the ordinary eigenvalues of the transformation $P^{-1}T$ or of the matrix $[p_{ij}]^{-1}[t_{ij}]$ -i.e., the solutions of the equation $\det [sp_{ij} - t_{ij}] = 0$.

Theorem 1: If $s_1 \leq \ldots \leq s_n$ are the generalized eigenvalues of T with respect to P, where T and P are symmetric transformations on E_n to E_n with domain E_n and P is positive definite, and if $\alpha_1,\ldots,\alpha_n$ is a P orthonormal basis of E_n such that $T\alpha_k = s_k P\alpha_k$ $(k = 1,\ldots,n)$,(1) then

$$s_i = \inf_{\alpha} \left\{(T\alpha,\alpha) \mid (P\alpha,\alpha) = 1,\ (P\alpha,\alpha_j) = 0\ (j = 1,\ldots,i-1)\right\}. \quad (i=1,\ldots,n)$$

Proof: Any vector α in E_n has the form

$$\alpha = x_1\alpha_1 + \ldots + x_n\alpha_n .$$

If the vector α is such that

$$(P\alpha,\alpha_j) = 0 \qquad (j = 1,\ldots,i-1),$$

then

$$0 = (P\alpha,\alpha_j) = (P(x_1\alpha_1 + \ldots + x_n\alpha_n),\alpha_j)$$

$$= x_1(P\alpha_1,\alpha_j) + \ldots + x_j(P\alpha_j,\alpha_j) + \ldots + x_n(P\alpha_n,a_j)$$

$$= x_j . \qquad (j = 1,\ldots,i-1)$$

Thus

$$\alpha = x_i\alpha_i + \ldots + x_n\alpha_n ,$$

(1) For a proof that such a P orthonormal basis exists, see Indritz, Methods in Analysis, p. 119.

and the requirement that $(P\alpha,\alpha) = 1$ implies that

$$x_i^2 + \dots + x_n^2 = 1 .$$

Now note that

$$
\begin{aligned}
(T\alpha,\alpha) &= (T(x_i\alpha_i + \dots + x_n\alpha_n),\ x_i\alpha_i + \dots + x_n\alpha_n) \\
&= (x_iT\alpha_i + \dots + x_nT\alpha_n, x_i\alpha_i + \dots + x_n\alpha_n) \\
&= (x_is_iP\alpha_i + \dots + x_ns_nP\alpha_n, x_i\alpha_i + \dots + x_n\alpha_n) \\
&= s_ix_i^2 + \dots + s_nx_n^2 \geq s_i(x_i^2 + \dots + x_n^2) = s_i .
\end{aligned}
$$

Hence

$$\inf_{\alpha} \left\{(T\alpha,\alpha) \,|\, (P\alpha,\alpha) = 1,\ (P\alpha,\alpha_j) = 0\ (j = 1, \dots, i-1)\right\} \geq s_i .$$

Now let $\alpha = \alpha_i$. Then

$$(P\alpha_i,\alpha_i) = 1 ,$$

$$(P\alpha_i,\alpha_j) = 0 \qquad (j = 1, \dots, i-1), \quad \text{and}$$

$$(T\alpha_i,\alpha_i) = (s_iP\alpha_i,\alpha_i) = s_i ,$$

since $\alpha_1, \dots, \alpha_n$ is a P orthonormal basis of generalized eigenvectors. Thus

$$\inf_{\alpha} \left\{(T\alpha,\alpha) \,|\, (P\alpha,\alpha) = 1,\ (P\alpha,\alpha_j) = 0\ \ (j = 1, \dots, i-1)\right\} \leq s_i .$$

A comparison of the two inequalities yields

$$s_i = \inf_{\alpha} \left\{(T\alpha,\alpha) \,|\, (P\alpha,\alpha) = 1,\ \ (P\alpha,a_j) = 0\ \ (j = 1, \dots, i-1)\right\} . \qquad \blacksquare$$

In order to calculate the generalized eigenvalues by use of Theorem 1, it is necessary to know the eigenvectors $\alpha_1,\ldots,\alpha_n$. In the next theorem this requirement is avoided.

Theorem 2: If $s_1 \leq \ldots \leq s_n$ are the generalized eigenvalues of T with respect to P, where T and P are symmetric transformations on E_n to E_n with domain E_n and P is positive definite, then

$$s_i = \sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right],\ (i=1,\ldots,n)$$

where the supremum is taken with respect to all sets $\beta_1,\ldots,\beta_{i-1}$ of $i-1$ vectors in E_n.

Proof: Let $\beta_1,\ldots,\beta_{i-1}$ be an arbitrary but fixed set of vectors in E_n. For this fixed set of $i-1$ vectors, let $\alpha = x_1\alpha_1+\ldots+x_i\alpha_i$ be a vector in E_n such that $(P\alpha,\beta_j) = 0$ $(j = 1,\ldots,i-1)$ and $(P\alpha,\alpha) = x_1^2+\ldots+x_i^2 = 1$. That such a choice is possible follows from the fact that $i-1$ homogeneous linear equations in i unknowns always have a nontrivial solution, which can be normalized.

For this choice of α,

$$\begin{aligned}(T\alpha,\alpha) &= (T(x_1\alpha_1+\ldots+x_i\alpha_i),\ x_1\alpha_1+\ldots+x_i\alpha_i)\\ &= (x_1s_1P\alpha_1+\ldots+x_is_iP\alpha_i, x_1\alpha_1+\ldots+x_i\alpha_i)\\ &= s_1x_1^2+\ldots+s_ix_i^2 \leq s_i(x_1^2+\ldots+x_i^2) = s_i.\end{aligned}$$

Hence

$$\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\} \leq s_i\ ;$$

and since this inequality is true for any set of vectors $\beta_1,\ldots,\beta_{i-1}$,

$$\sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right] \le s_i .$$

Now choose $\beta_j = \alpha_j$ $(j=1,\ldots,i-1)$, where $\alpha_1,\ldots,\alpha_n$ is the P orthonormal basis of eigenvectors previously used. Then from Theorem 1

$$\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\alpha_j)=0\ (j=1,\ldots,i-1)\right\} \ge s_i ,$$

which implies that

$$\sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right] \ge s_i .$$

Hence

$$\sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right] = s_i . \quad \blacksquare$$

<u>Definition 2</u>: Let α be a vector in E_n whose representation in the δ-basis (see Definition 1) is

$$\alpha \overset{\delta}{\longleftrightarrow} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} .$$

Suppose that the quadratic forms $(T\alpha,\alpha)$ and $(P\alpha,\alpha)$ are subjected to r $(< n)$ independent homogeneous constraints of the form

$$\sum_{k=1}^{n} a_{ki}x_i = 0 \quad (k=1,\ldots,r) ,$$

where the a_{ki} are real constants; that is, some r of the unknowns are eliminated from $(T\alpha,\alpha)$ and $(P\alpha,\alpha)$ by solving the equations of constraint for these r unknowns in terms of the remaining $n-r$.[(1)] The resulting forms in $n-r$ unknowns are called the reduced quadratic forms and are designated as $(T'\alpha',\alpha')$ and $(P'\alpha',\alpha')$. Here α' is a vector in E'_{n-r}, an $(n-r)$-dimensional real Euclidean vector space with orthonormal basis $\delta'_1,\ldots,\delta'_{n-r}$, where the j^{th} coordinate of δ'_j $(j=1,\ldots,n-r)$ is 1 and the other coordinates are zero. T' and P' are symmetric transformations on E'_{n-r} whose representations with respect to the δ' basis are the symmetric matrices $[t'_{ij}]$ and $[p'_{ij}]$. Note: the positive definiteness of P implies that P' also is positive definite.

A question that arises at this point is whether or not there is some relation between the generalized eigenvalues of T with respect to P and the generalized eigenvalues of T' with respect to P'. This question is answered in the following theorem.

Theorem 3: If $s_1 \leq \ldots \leq s_n$ are the generalized eigenvalues of T with respect to P, and if $s'_1 \leq \ldots \leq s'_{n-r}$ are the generalized eigenvalues of T' with respect to P', then $s_i \leq s'_i \leq s_{i+r}$ $(i=1,\ldots,n-r)$, where r $(< n)$ is the number of independent constraints referred to in Definition 2.

---

(1) There is no loss of generality in supposing that it is possible to solve the equations of constraint for the last r unknowns in terms of the first $n-r$; for if this were not initially the case, it could be made so simply by relabeling the unknowns. It is therefore assumed here and subsequently that the equations of constraint are used to express $x_{n-r+1},\ldots,x_n$ in terms of $x_1,\ldots,x_{n-r}$.

In preparation for a proof of this theorem, an observation and two lemmas are needed.

Observation: The constraints $\sum_{k=1}^{n} a_{ki}x_i = 0$ $(k=1,\ldots,r)$ can be expressed in the form $(P\alpha,\gamma_k) = 0$ $(k=1,\ldots,r)$, where α is a vector in E_n and the γ_k are r linearly independent vectors in E_n. For if $\alpha = x_1\delta_1+\ldots+x_n\delta_n$ and $P\gamma_k = a_{k1}\delta_1+\ldots+a_{kn}\delta_n$ $(k=1,\ldots,r)$, then

$$(\alpha,P\gamma_k) = \sum_{k=1}^{n} a_{ki}x_i .$$

But P is symmetric. So

$$(\alpha,P\gamma_k) = (P\alpha,\gamma_k) .$$

Hence

$$(P\alpha,\gamma_k) = \sum_{k=1}^{n} a_{ki}x_i = 0 \quad (k=1,\ldots,r).$$

Lemma 1: If u is a vector in E_n having the representation

$$u \overset{\delta}{\longleftrightarrow} \begin{bmatrix} u_1 \\ \circ \\ \circ \\ \circ \\ u_n \end{bmatrix}, \tag{A}$$

define u' to be that vector in E'_{n-r} which has the representation

$$u' \overset{\delta'}{\longleftrightarrow} \begin{bmatrix} u_1 \\ \circ \\ \circ \\ \circ \\ u_{n-r} \end{bmatrix}; \tag{B}$$

and if u' is a vector in E'_{n-r} having the representation (B), let u

be the vector in E_n with representation (A), where the coordinates $u_{n-r+1},\dots,u_n$ are determined by use of the equations $(Pu,\gamma_k) = 0$ $(k=1,\dots,r)$. Now if α and β are vectors in E_n such that

$$(P\alpha,\gamma_k) = 0 \quad \text{and} \quad (P\beta,\gamma_k) = 0, \qquad (k=1,\dots,r)$$

then

$$(P'\alpha',\beta') = (P\alpha,\beta) .$$

Proof of Lemma 1: Recall (see Definition 2) that the transformation P' on E'_{n-r} is so defined that

$$(P'u',u') = (Pu,u)$$

for every u' and corresponding u for which $(Pu,\gamma_k) = 0$ $(k=1,\dots,r)$. Now suppose that

$$(P\alpha,\gamma_k) = 0 \quad (k=1,\dots,r)$$

and

$$(P\beta,\gamma_k) = 0. \quad (k=1,\dots,r)$$

Then

$$(P(\alpha-\beta),\gamma_k) = 0; \quad (k=1,\dots,r)$$

so

$$(P'(\alpha'-\beta'),\alpha'-\beta') = (P(\alpha-\beta),\alpha-\beta).$$

Expanding both sides yields

$$-(P'\alpha',\beta') - (P'\beta',\alpha') = -(P\alpha,\beta) - (P\beta,\alpha) ,$$

where use has been made of the fact that $(P'\alpha',\alpha') = (P\alpha,\alpha)$ and $(P'\beta',\beta') = (P\beta,\beta)$. Since P and P' are symmetric and since the inner product is symmetric,

$$(P'\beta',\alpha') = (\beta',P'\alpha') = (P'\alpha',\beta')$$

and

$$(P\beta,\alpha) = (\beta,P\alpha) = (P\alpha,\beta) .$$

Thus,

$$-2\ (P'\alpha',\beta') = -2\ (P\alpha,\beta),$$

or

$$(P'\alpha',\beta') = (P\alpha,\beta) .$$

Lemma 2:

$$\sup_{\beta'}\left[\inf_{\alpha'}\left\{(T'\alpha',\alpha') \mid (P'\alpha',\alpha') = 1,\ (P'\alpha',\beta'_j) = 0\ (j=1,\ldots,i-1)\right\}\right] =$$
$$\sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha) \mid (P\alpha,\alpha) = 1, (P\alpha,\gamma_k)=0\ (k=1,\ldots,r),\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right],$$
$$(i=1,\ldots,n-r)$$

the suprema being taken respectively over every set of $i-1$ vectors β'_j in E'_{n-r} and every set of $i-1$ vectors β_j in E_n. Here

$$(T'\alpha',\alpha) = \left[(T\alpha,\alpha) \mid (P\alpha,\gamma_k) = 0 \quad (k=1,\ldots,r)\right]$$

and

$$(P'\alpha',\alpha') = \left[(P\alpha,\alpha) \mid (P\alpha,\gamma_k) = 0\ (k=1,\ldots,r)\right] .$$

Proof of Lemma 2: The proof consists of showing first that

$$\sup_{\beta'} \leq \sup_{\beta} ,$$

then that

$$\sup_{\beta'} \geq \sup_{\beta} ,$$

and hence that the suprema are equal.

Let β'_j $(j=1,\ldots.i-1<n-r)$ be an arbitrarily chosen but fixed set of vectors in E'_{n-r}. Construct a set of vectors β_j $(j=1,\ldots,i-1)$ in E_n according to the correspondence described in Lemma 1; that is, for each j, the first $n-r$ coordinates of β_j are the same as those of β'_j, and the last r coordinates of β_j are calculated from the equations $(P\beta_j,\gamma_k) = 0$ $(k=1,\ldots,r)$. Now let α be any vector in E_n such that $(P\alpha,\alpha) = 1$, $(P\alpha,\gamma_k) = 0$ $(k=1,\ldots,r)$, and $(P\alpha,\beta_j) = 0$ $(j=1,\ldots,i-1)$; and let α' be the corresponding vector in E'_{n-r}. It then follows from Lemma 1 that

$$(P'\alpha',\beta'_j) = 0 \quad (j=1,\ldots,i-1)$$

and from the definitions of P' and T' that

$$(P'\alpha',\alpha') = 1$$

and

$$(T'\alpha',\alpha') = (T\alpha,\alpha) .$$

That is, for the given fixed set of β'_j and the fixed set of β_j constructed as described, to any α in E_n satisfying the conditions $(P\alpha,\alpha) = 1$, $(P\alpha,\gamma_k) = 0$ $(k=1,\ldots,r)$, $(P\alpha,\beta_j) = 0$ $(j=1,\ldots,i-1)$ the corresponding α' in E'_{n-r} is such that $(P'\alpha',\alpha') = 1$, $(P'\alpha',\beta'_j) = 0$ $(j=1,\ldots,i-1)$, and $(T'\alpha',\alpha')$ <u>is the same number as</u> $(T\alpha,\alpha)$. Hence,

$$\inf_{\alpha}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\beta'_j)=0\ (j=1,\ldots,i-1)\right\} =$$

$$\inf_{\alpha}\Big\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_k)=0\ (k=1,\ldots,r),$$

$$(P\alpha,\beta_j)=0\ (j=1,\ldots,i-1),\ (P\beta_j,\gamma_k)=0\ (j=1,\ldots,i-1;k=1,\ldots,r)\Big\}.$$

$$(i=1,\ldots,n-r)$$

Since any α in E_n such that $(P\alpha,\gamma_k) = 0$ $(k=1,\ldots,r)$ can be obtained by a suitable choice of an α' in E'_{n-r} (and conversely), the infimum on the left may be taken with respect to α' instead of α. Thus,

$$\inf_{\alpha'}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\beta'_j)=0\ (j=1,\ldots,i-1)\right\} =$$

$$\inf_{\alpha}\Big\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_k)=0\ (k=1,\ldots,r),\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1),$$

$$(P\beta_j,\gamma_k)=0\ (j=1,\ldots,i-1;k=1,\ldots,r)\Big\}.$$

$$(i=1,\ldots,n-r)$$

Since this relation holds for any set of $i-1$ vectors β'_j in E'_{n-r} and corresponding set of $i-1$ vectors β_j in E_n,

$$\sup_{\beta'}\left[\inf_{\alpha'}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\beta'_j)=0\ (j=1,\ldots,i-1)\right\}\right] =$$

$$\sup_{\beta'}\Big[\inf_{\alpha}\Big\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_k)=0\ (k=1,\ldots,r),\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1),$$

$$(P\beta_j,\gamma_k)=0\ (j=1,\ldots,i-1;k=1,\ldots,r)\Big\}\Big];$$

$$(i=1,\ldots,n-r)$$

and since any vector β'_j in E'_{n-r} can be obtained by a suitable choice of a vector β_j in E_n such that $(P\beta_j,\gamma_k) = 0$, j fixed, $k=1,\ldots,r$

(and conversely), the supremum on the right side may be taken with respect to the sets β_j $(j=1,\ldots,i-1)$ instead of the sets β'_j $(j=1,\ldots,i-1)$. Hence,

$$\sup_{\beta'}\left[\inf_{\alpha'}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\beta'_j)=0\ (j=1,\ldots,i-1)\right\}\right] =$$

$$\sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_k)=0\ (k=1,\ldots,r),\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1),\right.\right.$$
$$\left.\left.(P\beta_j,\gamma_k)=0\ (j=1,\ldots,i-1;k=1,\ldots,r)\right\}\right].$$
$$(i=1,\ldots,n-r)$$

If the domain from which the β_j may be selected is extended by deleting the restrictions $(P\beta_j,\gamma_k) = 0$ $(j=1,\ldots,i-1;k=1,\ldots,r)$, the supremum on the right either remains the same or increases. Therefore

$$\sup_{\beta'}\left[\inf_{\alpha'}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\beta'_j)=0\ (j=1,\ldots,i-1)\right\}\right] \leq$$
$$\sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,(P\alpha,\gamma_k)=0\,(k=1,\ldots,r),\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right]. \qquad (C)$$
$$(i=1,\ldots,n-r)$$

To show that the inequality (C) holds with the sense reversed, it is convenient to use a basis of E_n constructed as follows. Let the first $n-r$ vectors of the basis (call them $t_1,\ldots,t_{n-r}$) correspond (in the sense of Lemma 1) to the vectors of the δ' basis in E'_{n-r}. That is, t_i $(i=1,\ldots,n-r)$ has 1 as its i^{th} coordinate; the remainder of its first $n-r$ coordinates are zero; and the last r coordinates are calculated from the first $n-r$ by using the equations

$$(Pt_i,\gamma_k) = 0. \qquad (i \ \text{fixed};\ k=1,\ldots,r)$$

The $n-r$ vectors t_i $(i=1,\ldots,n-r)$ so constructed are linearly independent; and since each of them is P orthogonal to every γ_k $(k=1,\ldots,r)$, the n vectors

$$t_1,\ldots,t_{n-r}, \gamma_1,\ldots,\gamma_r$$

are linearly independent. These n vectors constitute the desired basis of E_n.

Now let β_j $(j=1,\ldots,i-1)$ be an arbitrarily chosen but fixed set of $i-1$ vectors in E_n. For each fixed j, β_j can be written in the form

$$\beta_j = \hat{\beta}_j + \tilde{\beta}_j ,$$

where

$$\hat{\beta}_j = c_{j1}\gamma_1+\ldots+c_{jr}\gamma_r$$

and

$$\tilde{\beta}_j = d_{j1}t_1+\ldots+d_{j,n-r}t_{n-r}$$

for suitably chosen constants $c_{j1},\ldots,c_{jr},d_{j1},\ldots,d_{j,n-r}$. The conditions $(P\alpha,\beta_j) = 0$ $(j=1,\ldots,i-1)$ are equivalent to

$$(P\alpha,\hat{\beta}_j+\tilde{\beta}_j) = (P\alpha,\hat{\beta}_j) + (P\alpha,\tilde{\beta}_j) = (P\alpha,\tilde{\beta}_j) = 0 ,$$

since $\hat{\beta}_j$ is a linear combination of the vectors γ_k, for which $(P\alpha,\gamma_k) = 0$ $(k=1,\ldots,r)$; and furthermore, $(P\tilde{\beta}_j,\gamma_k) = 0$ $(j=1,\ldots,i-1;$ $k=1,\ldots,r)$ since $\tilde{\beta}_j$ is a linear combination of the vectors t_i $(i=1,\ldots,n-r)$, for which $(Pt_i,\gamma_k) = 0$ $(i=1,\ldots,n-r;k=1,\ldots,r)$. With the $\tilde{\beta}_j$ represented in the δ basis, construct a set of vectors $\tilde{\beta}'_j$ $(j=1,\ldots,i-1)$ in E'_{n-r} according to the correspondence described in Lemma 1. Let α be any vector in E_n such that $(P\alpha,\alpha) = 1$, $(P\alpha,\gamma_k) = 0$ $(k=1,\ldots,r)$, and $(P\alpha,\beta_j) = (P\alpha,\tilde{\beta}_j) = 0$ $(j=1,\ldots,i-1)$; and let α' be the corresponding vector in E'_{n-r}. It then follows as before by use of Lemma 1 that $(P'\alpha',\tilde{\beta}'_j) = 0$ $(j=1,\ldots,i-1)$ and from the definitions of P' and T' that $(P'\alpha',\alpha') = 1$ and $(T'\alpha',\alpha') = (T\alpha,\alpha)$. Hence, as before,

$$\inf_{\alpha'}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\tilde{\beta}'_j)=0\ (j=1,\ldots,i-1)\right\} =$$
$$\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_k)=0\ (k=1,\ldots,r),\ (P\alpha,\beta_j)=(P\alpha,\tilde{\beta}_j)=0\ (j=1,\ldots,i-1)\right\}.$$
$$(i=1,\ldots,n-r)$$

Since this relation holds for any set of $i-1$ vectors β_j in E_n provided the $i-1$ vectors $\tilde{\beta}'_j$ in E'_{n-r} are constructed as described,

$$\sup_{\beta}\left[\inf_{\alpha'}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\tilde{\beta}'_j)=0\ (j=1,\ldots,i-1)\right\}\right] =$$
$$\sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,(P\alpha,\gamma_k)=0\ (k=1,\ldots,r),\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right].$$
$$(i=1,\ldots,n-r)$$

Since for each fixed j the choice of a β_j in E_n leads to a unique $\tilde{\beta}'_j$ in E'_{n-r}, the supremum on the left side may be taken with respect to sets of $i-1$ vectors $\tilde{\beta}'_j$ which are images (under the correspondence described in Lemma 1) of sets of $i-1$ vectors $\tilde{\beta}_j$ in E_n such that $(P\tilde{\beta}_j,\gamma_k) = 0$ $(j=1,\ldots,i-1;k=1,\ldots,r)$. Thus,

$$\sup_{\tilde{\beta}'}\left[\inf_{\alpha'}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\tilde{\beta}'_j)=0\ (j=1,\ldots,i-1)\right\}\right] =$$
$$\sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_k)=0\ (k=1,\ldots,r),\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right].$$
$$(i=1,\ldots,n-r)$$

If the restriction that the vectors $\tilde{\beta}'_j$ be images of vectors $\tilde{\beta}_j$ such that $(P\tilde{\beta}_j,\gamma_k) = 0$ is removed, the supremum on the left either remains the same or increases. Hence,

$$\sup_{\beta'}\left[\inf_{\alpha'}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\beta'_j)=0\ (j=1,\ldots,i-1)\right\}\right] \geq$$
$$\sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_k)=0\ (k=1,\ldots,r),\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right]. \quad \text{(D)}$$
$$(i=1,\ldots,n-r)$$

From the inequalities (C) and (D) the desired conclusion of Lemma 2 follows.

Proof of Theorem 3: Theorem 2 yields

$$s_i = \sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right]; \quad (i=1,\ldots,n) \quad \text{(E)}$$

$$s'_i = \sup_{\beta'}\left[\inf_{\alpha'}\left\{(T\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\beta'_j)=0\ (j=1,\ldots,i-1)\right\}\right]. \quad \text{(F)}$$
$$(i=1,\ldots,n-r)$$

A comparison of (F) and Lemma 2 yields

$$s'_i = \sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_k)=0\ (k=1,\ldots,r),(P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right].$$
$$(i=1,\ldots,n-r)$$

Let $\beta_1,\ldots,\beta_{i-1}$ be any fixed set of $i-1$ vectors in E_n. Then

$$\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\} \leq$$
$$\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_k)=0\ (k=1,\ldots,r),\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\},$$

since the infimum on the right is taken with respect to a set of vectors

α in E_n restricted by the conditions $(P\alpha,\gamma_k) = 0$ $(k=1,\ldots,r)$, while no such restrictions apply to the α's on the left.

The result

$$s_i \leq s'_i$$

follows immediately by taking the supremum of both sides of the inequality over all sets of $i-1$ vectors β_j $(j=1,\ldots,i-1)$ in E_n.

Now consider the relation

$$s_{i+r} = \sup_{\beta} \left[\inf_{\alpha} \left\{ (T\alpha,\alpha) \mid (P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1+r) \right\} \right] . \qquad (i=1,\ldots,n-r)$$

If the sets of $i-1+r$ vectors β_j are so chosen that $\beta_j = \gamma_j$ $(j=1,\ldots,r)$ while β_j is arbitrary for $j=r+1,\ldots,i-1+r$, then

$$\sup_{\bar{\beta}} \left[\inf_{\alpha} \left\{ (T\alpha,\alpha) \mid (P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1+r) \right\} \right] \leq s_{i+r} ,$$

since the supremum is taken over the class $\bar{\beta}$ of sets of $i-1+r$ vectors which are restricted by the description just given. But the supremum on the left is exactly s'_i. Hence

$$s'_i \leq s_{i+r} . \qquad (i=1,\ldots,n-r) . \qquad \blacksquare$$

<u>Definition 3</u>: If $K(x_1,\ldots,x_n)$ is a quadratic form in n unknowns $x_1,\ldots,x_n$, then the quadratic form $K'(x_1,\ldots,x_{n-1})$ obtained from K by setting $x_n = 0$ is called the $(n-1)$st section of K.

<u>Corollary to Theorem 3</u>: If K has real coefficients and if the eigenvalues of K and K' are arranged in increasing order

$$s_1 \leq \cdots \leq s_n \qquad \text{and} \qquad s'_1 \leq \cdots \leq s'_{n-1} ,$$

then

$$s_i \leq s'_i \leq s_{i+1} . \quad (i=1,\ldots,n-1)$$

<u>Proof of Corollary</u>: In Theorem 3 let $r=1$, and let the constraint have the form $x_n = 0$. Identify $(T\alpha,\alpha)$ with $K(x_1,\ldots,x_n)$, and let $(P\alpha,\alpha) = x_1^2+\cdots+x_n^2$.

CHAPTER III

THEOREMS CONCERNING THE NATURAL FREQUENCIES OF LINEAR SYSTEMS

In this chapter the basic mathematical theorems of Chapter II are used to derive four additional theorems describing the natural frequencies of linear systems. The statements of these additional theorems are copied essentially word for word from Courant-Hilbert (page 286). Theorems 4 and 5 describe the relation that exists between the natural frequencies of a linear system and the natural frequencies of a new system obtained from the first by imposing r linear homogeneous constraints on the generalized coordinates. The last two theorems indicate what changes occur in the natural frequencies when a system is modified in such a way that either the potential energy increases and the kinetic energy remains the same or the kinetic energy increases and the potential energy remains the same for all values of the generalized coordinates.

The physical systems considered are loosely described as dissipationless linear systems of n degrees of freedom executing small free vibrations about a position of stable equilibrium. Mathematically, this terminology will be understood to mean a conservative physical system in which, in particular,

(1) the minimum number of geometric coordinates q_i required to specify the positions of all the inertial elements at any time is n;

(2) the kinetic energy is a positive definite quadratic form in the generalized velocities $\dot{q}_i$ with <u>constant</u> coefficients;

(3) the potential energy is a positive definite quadratic form in the generalized coordinates q_i also with constant coefficients.

Theorem 4: The p^{th} overtone is a vibrating system having n degrees of freedom is the highest of the fundamental tones of the systems obtained from the given system by imposing p independent arbitrarily chosen restrictions of the form $\sum_{i=1}^{n} a_{ji}x_i = 0$ $(j=1,\ldots,p)$, $p < n$ (the fundamental tone is the lowest of the natural frequencies and the p^{th} overtone is the $(p+1)^{st}$ natural frequency).

Proof: Let S denote the original system and S' the system obtained by putting p restrictions on S. Now define

$(P\alpha,\alpha)$ = kinetic energy of S;[1]

$(T\alpha,\alpha)$ = potential energy of S;

$(P'\alpha',\alpha')$ = kinetic energy of S';

$(T'\alpha',\alpha')$ = potential energy of S';

$\sqrt{s_i} = \nu_i$ = the i^{th} natural frequency of S,

where $\nu_1 \leq \ldots \leq \nu_n$;

$\sqrt{s'_i} = \nu'_i$ = the i^{th} natural frequency of S',

where $\nu'_1 \leq \ldots \leq \nu'_{n-p}$.

From Theorem 2

---

(1) Actually the kinetic energy should be identified with $(P\frac{d\alpha}{dt},\frac{d\alpha}{dt})$. But in the present discussion, which depends on the purely algebraic properties of the quadratic form $(P\alpha,\alpha)$, it is immaterial whether the indeterminate α represents the generalized coordinates or their time derivatives. Hence, for simplicity of notation, here and subsequently the kinetic energy is identified with $(P\alpha,\alpha)$ itself.

$$s_i = \sup_{\beta}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right] \text{ and}$$

$$s'_i = \sup_{\beta'}\left[\inf_{\alpha'}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha')=1,\ (P'\alpha',\beta'_j)=0\ (j=1,\ldots,i-1)\right\}\right].^{(1)}$$

Thus the square of the fundamental frequency of S' is

$$s'_1 = \inf_{\alpha'}\left\{(T'\alpha',\alpha')\,|\,(P'\alpha',\alpha') = 1\right\}.$$

However, since the constraints $\sum_{i=1}^{n} a_{ji}x_i = 0\ (j=1,\ldots,p)$ can be written in the form $(P\alpha,\gamma_j) = 0\ (j=1,\ldots,p)$,

$$(T'\alpha',\alpha') = \left[(T\alpha,\alpha)\,|\,(P\alpha,\gamma_j)=0\ (j=1,\ldots,p)\right]$$

and

$$(P'\alpha',\alpha') = \left[(P\alpha,\alpha)\,|\,(P\alpha,\gamma_j)=0\ (j=1,\ldots,p)\right].$$

This implies that

$$s'_1 = \inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_j)=0\ (j=1,\ldots,p)\right\}.$$

Of course S' depends on the constraints imposed on S, and thus the fundamental frequency of S' is dependent on these constraints. Hence the maximum of these fundamental frequencies is

$$\begin{aligned}\max v'_1 &= \sup_{\gamma}\sqrt{\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_j)=0\ (j=1,\ldots,p)\right\}}\\ &= \sqrt{\sup_{\gamma}\left[\inf_{\alpha}\left\{(T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\gamma_j)=0\ (j=1,\ldots,p)\right\}\right]}\\ &= \sqrt{s_{p+1}}\\ &= v_{p+1},\end{aligned}$$

---

(1) The squares of the natural frequencies are the generalized eigenvalues of T with respect to P (see Indritz, p. 122).

which is indeed the p^{th} overtone of S. ∎

Theorem 5: If by imposing r independent conditions of the form $\sum_{i=1}^{n} a_{ji}x_i = 0$ $(j=1,\ldots,r)$ on a system S an "r-fold restricted" system S' is obtained, then the frequencies $\nu'_1,\ldots,\nu'_{n-r}$ of the restricted system are neither smaller than the corresponding frequencies $\nu_1,\ldots,\nu_{n-r}$ nor larger than the frequencies $\nu_{r+1},\ldots,\nu_n$ of the free system; that is,

$$\nu_i \leq \nu'_i \leq \nu_{i+r} \quad (i=1,\ldots,n-r).$$

Proof: Define

$$(P\alpha,\alpha) = \text{kinetic energy of } S;$$

$$(T\alpha,\alpha) = \text{potential energy of } S;$$

$$(P'\alpha',\alpha') = \text{kinetic energy of } S';$$

$$(T'\alpha',\alpha') = \text{potential energy of } S';$$

$$\sqrt{s_i} = \nu_i \qquad \nu_1 \leq \cdots \leq \nu_n;$$

$$\sqrt{s'_i} = \nu'_i \qquad \nu'_1 \leq \cdots \leq \nu'_{n-r}.$$

Then s_i is the i^{th} generalized eigenvalue of T with respect to P and s'_i is the i^{th} generalized eigenvalue of T' with respect to P'. From Theorem 3

$$s_i \leq s'_i \leq s_{i+r}, \quad (i=1,\ldots,n-r)$$

Thus

$$\nu_i \leq \nu'_i \leq \nu_{i+r}. \quad (i=1,\ldots,n-r) \qquad ∎$$

Theorem 6: As the inertia increases, the pitch of the fundamental tone and every overtone decreases (or remains the same). Note: Increase of inertia means change to a system with kinetic energy $(P'\alpha,\alpha)$ such that for every α $(P'\alpha,\alpha) - (P\alpha,\alpha) \geq 0$, while the potential energy remains unchanged.

Proof: Let $(P\alpha,\alpha)$ be the kinetic energy of the original system S, $(P'\alpha,\alpha)$ the kinetic energy of the new system S', and $(T\alpha,\alpha)$ the potential energy for both systems. The squares of the natural frequencies of S and S' are the eigenvalues of the transformations $P^{-1}T$ and $(P')^{-1}T$ respectively.

If λ_i is the i^{th} eigenvalue of the transformation $T^{-1}P$ (note the interchange of roles of T and P) and λ_i' is the i^{th} eigenvalue of the transformation $T^{-1}P'$, where $\lambda_1 \leq \ldots \leq \lambda_n$ and $\lambda_1' \leq \ldots \leq \lambda_n'$, then from Theorem 2

$$\lambda_i = \sup_{\beta} \left[\inf_{\alpha} \left\{ (P\alpha,\alpha) \mid (T\alpha,\alpha)=1,\ (T\alpha,\beta_j)=0\ (j=1,\ldots,i-1) \right\} \right]$$

and

$$\lambda_i' = \sup_{\beta} \left[\inf_{\alpha} \left\{ (P'\alpha,\alpha) \mid (T\alpha,\alpha)=1,\ (T\alpha,\beta_j)=0\ (j=1,\ldots,i-1) \right\} \right].$$

Recall that $(P'\alpha,\alpha) \geq (P\alpha,\alpha)$ for every α. This implies that for any fixed set of $i-1$ vectors $\beta_1,\ldots,\beta_{i-1}$ in E_n

$$\inf_{\alpha} \left\{ (P'\alpha,\alpha) \mid (T\alpha,\alpha)=1,\ (T\alpha,\beta_j)=0\ (j=1,\ldots,i-1) \right\} \geq$$

$$\inf_{\alpha} \left\{ (P\alpha,\alpha) \mid (T\alpha,\alpha)=1,\ (T\alpha,\beta_j)=0\ (j=1,\ldots,i-1) \right\}.$$

However, this is true for each arbitrary choice of $i-1$ vectors

$\beta_1,\ldots,\beta_{i-1}$. Thus

$$\sup_{\beta}\left[\inf_{\alpha}\left\{(P'\alpha,\alpha)\,|\,(T\alpha,\alpha)=1,\ (T\alpha,\beta_j)=0\ (j=1,\ldots,i-1)\right\}\right] \geq$$

$$\sup_{\beta}\left[\inf_{\alpha}\left\{(P\alpha,\alpha)\,|\,(T\alpha,\alpha)=1,\ (T\alpha,\beta_j)=0\ \ (j=1,\ldots,i-1)\right\}\right] ,$$

or

$$\lambda_i' \geq \lambda_i \qquad (i=1,\ldots,n) .$$

Now note that if λ is an eigenvalue of $T^{-1}P$, then $1/\lambda$ is an eigenvalue of $P^{-1}T$; for if

$$T^{-1}P\alpha = \lambda\alpha ,$$

then

$$(P^{-1}T)(T^{-1}P\alpha) = (P^{-1}T)\lambda\alpha ,$$

or

$$1/\lambda\alpha = (P^{-1}T)\alpha .$$

Thus if s_i is the i^{th} eigenvalue of $P^{-1}T$, where $s_1 \geq \cdots \geq s_n$, then

$$s_i = 1/\lambda_i \qquad (i=1,\ldots,n).$$

Similarly

$$s_i' = 1/\lambda_i' \qquad (i=1,\ldots,n) ,$$

where s_i' is the i^{th} eigenvalue of the transformation $(P')^{-1}T$. Hence

$$s_i' \leq s_i \qquad (i=1,\ldots,n) .$$

The desired result is obtained immediately since s_i and s_i' are the squares of the natural frequencies for the systems S and S' respectively. ∎

<u>Theorem 7</u>: If the system stiffens, the pitch of the fundamental tone and every overtone increases (or remains the same). <u>Note</u>: Stiffening of the system means change to a system whose kinetic energy is the same but whose potential energy is the same or greater for the same values of the coordinates.

<u>Proof</u>: Define S to be the original system with potential energy $(T\alpha,\alpha)$ and S' to be the new system with potential energy $(T'\alpha,\alpha)$. Let $(P\alpha,\alpha)$ be the kinetic energy for both systems.

The squares of the natural frequencies of S and S' are the eigenvalues of the transformations $P^{-1}T$ and $P^{-1}T'$ respectively. Define s_i to be the i^{th} eigenvalue of $P^{-1}T$ and s_i' to be the i^{th} eigenvalue of $P^{-1}T'$. If the s_i and s_i' are so ordered that

$$s_1 \leq \cdots \leq s_n$$

and

$$s_1' \leq \cdots \leq s_n' ,$$

then from Theorem 2

$$s_i = \sup_{\beta} \left[\inf_{\alpha} \left\{ (T\alpha,\alpha) \,\middle|\, (P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1) \right\} \right]$$

and

$$s_i' = \sup_{\beta} \left[\inf_{\alpha} \left\{ (T'\alpha,\alpha) \,\middle|\, (P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\ldots,i-1) \right\} \right] .$$

However,

$$(T'\alpha,\alpha) \geq (T\alpha,\alpha) \quad \text{for all} \quad \alpha;$$

so

$$\sup_{\beta}\left[\inf_{\alpha}\left\{(T'\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ (P\alpha,\beta_j)=0\ (j=1,\dots,i-1)\right\}\right] \geq$$

$$\sup_{\beta}\left[\inf_{\alpha}\ (T\alpha,\alpha)\,|\,(P\alpha,\alpha)=1,\ \ (P\alpha,\beta_j)=0\ (j=1,\dots,i-1)\right\}\right] .$$

Thus

$$\sqrt{s_i'} \geq \sqrt{s_i} \quad (i=1,\dots,n).$$ ∎

CHAPTER IV

NUMERICAL ILLUSTRATIONS

An examination of the four theorems in Chapter III reveals that Theorem 4 is essentially a mathematical theorem and that Theorems 5, 6, and 7 are more closely associated with the comparison of the natural frequencies of physical systems. The purpose of this chapter is to illustrate Theorems 5, 6, and 7 by exhibiting the natural frequencies of some simple mechanical systems.

The following physical assumptions are made regarding the mechanical systems that are involved in the following examples.

(1) All springs are massless, are linear, and have spring constant k.

(2) All motion is in a straight line.

(3) All systems are frictionless.

A Result of Stiffening a System

Let S and S' represent the physical systems in Figures 1(a) and 1(b), respectively, and let w_i and w'_i $(i=1,\ldots,n)$ denote the respective natural frequencies, where

$$w_1 \leq \cdots \leq w_n$$

and

$$w'_1 \leq \cdots \leq w'_n .$$

A comparison of the kinetic and potential energies of the two systems

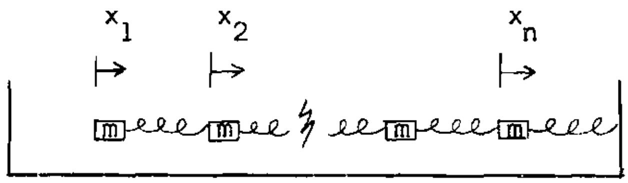

Figure 1(a)

The Original System S

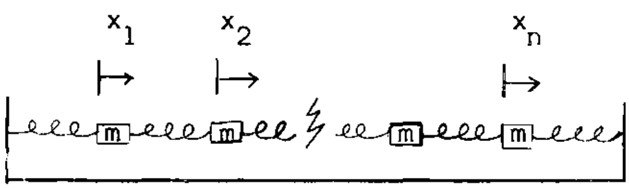

Figure 1(b)

The Stiffened System S'

Figure 1. An Example of Stiffening a System.

reveals that the systems have the same kinetic energy for any common set of values of the generalized velocities but that the potential energy of S' is greater than or equal to the potential energy of S for any common set of generalized coordinates. Thus the hypotheses of Theorem 7 are satisfied, which implies that

$$w_i' \geq w_i \qquad (i=1,\ldots,n) .$$

To verify that this relation does in fact hold in the special case under consideration, the natural frequencies of the two systems will be displayed and then compared.

It is well known that

$$w_i = 2\sqrt{k/m} \quad \sin\left[\left(\frac{2i-1}{2n+1}\right) \pi/2\right] \quad (i=1,\ldots,n)$$

and

$$w_i' = 2\sqrt{k/m} \quad \sin\left[\left(\frac{i}{n+1}\right) \pi/2\right] \quad (i=1,\ldots,n) .$$

Since

$$\frac{i}{n+1} \geq \frac{2i-1}{2n+1} \qquad (i=1,\ldots,n)$$

and since the sine function is an increasing function on the interval $(0, \pi/2)$,

$$w_i' \geq w_i \qquad (i=1,\ldots,n) .$$

The Effect of Increasing the Inertia

Denote the systems in Figures 2(a) and 2(b) by S and S', respectively, and let w_i and w_i' represent the corresponding natural frequencies, where

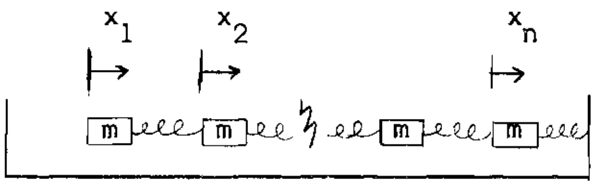

Figure 2(a)

The Original System S

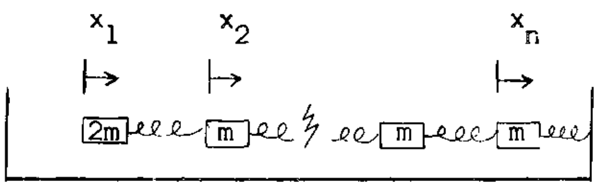

Figure 2(b)

The System S' with Increased Inertia

Figure 2. An Example of Increasing the Inertia of a System.

$$w_1 \leq \dots \leq w_n$$

and

$$w'_1 \leq \dots \leq w'_n .$$

An inspection of systems S and S' reveals that the potential energy is the same for both systems for any set of generalized coordinates. However, the kinetic energy of S' is greater than or equal to the kinetic energy of S for any common set of generalized velocities. Thus the hypotheses of Theorem 6 are satisfied, which implies that

$$w'_i \leq w_i \quad (i=1,\dots,n) .$$

This result will now be verified by direct computation.

The natural frequencies for S are

$$w_i = 2\sqrt{\frac{k}{m}} \sin \left[\left(\frac{2i-1}{2n+1}\right) \pi/2\right] \; (i=1,\dots,n).$$

An investigation of the natural frequencies of S' will now be conducted.

The equations of motion for S' are

$$\begin{aligned} 2m\ddot{x}_1 &= k(x_2 - x_1); \\ m\ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2); \\ &\vdots \\ m\ddot{x}_n &= -k(x_n - x_{n-1}) - kx_n . \end{aligned}$$

Now seek a solution of the form

$$x_i = A_i \sin wt .$$

Upon substituting into the equations of motion, it becomes apparent that the only non-trivial solutions occur when w satisfies the equation

$$\begin{vmatrix} (1 - 2\frac{m}{k} w^2) & -1 & & \bigcirc \\ -1 & (2 - \frac{m}{k} w^2) & -1 & \\ & \ddots & \ddots & \\ \bigcirc & & -1 & (2 - \frac{m}{k} w^2) \end{vmatrix} = 0 .$$

Let $\lambda = \frac{m}{k} w^2$ and define

$$D_n(\lambda) = \begin{vmatrix} (1 - 2\lambda) & -1 & & \bigcirc \\ -1 & (2 - \lambda) & -1 & \\ & \ddots & \ddots & \\ \bigcirc & & -1 & (2 - \lambda) \end{vmatrix}$$

It follows by an easy calculation that

$$D_{n+1}(\lambda) - (2 - \lambda) D_n(\lambda) + D_{n-1}(\lambda) = 0 ,$$

where

$$D_0(\lambda) = 1$$

and

$$D_1(\lambda) = 1 - 2\lambda .$$

Now let

$$\cos \alpha = \frac{2 - \lambda}{2} .$$

The difference equation with initial conditions becomes

$$y_{n+1}(\alpha) - 2\cos\alpha y_n(\alpha) + y_{n-1}(\alpha) = 0;$$
$$y_o(\alpha) = 1;$$
$$y_1(\alpha) = 4\cos\alpha - 3.$$

The solution for this difference equation with initial conditions is

$$y_n(\alpha) = \frac{2\cos(n\alpha + \alpha/2) - \cos(n\alpha - \alpha/2)}{\cos\alpha/2}.$$

This implies that the natural frequencies of S' are

$$w_i' = 2\sqrt{k/m}\ \sin\left(\frac{\alpha_i'}{2}\right) \quad (i=1,\ldots,n),$$

where $\alpha_1',\ldots,\alpha_n'$ $(0 < \alpha_i' < \pi,\ i=1,\ldots,n)$ are solutions to the equation

$$\frac{2\cos(n\alpha + \alpha/2) - \cos(n\alpha - \alpha/2)}{\cos\alpha/2} = 0,$$

that is, zeros of the continuous function $y_n(\alpha)$. A crucial bit of information about the location of these zeros may be gained by examining the sign of $y_n(\alpha)$ at $n+1$ successive points $\alpha = 0$ and $\alpha = \alpha_i$ $(i=1,\ldots,n)$, where

$$\alpha_i = \frac{2i - 1}{2n = 1}\pi \quad (i=1,\ldots,n).$$

Thus

$$y_n(0) > 0,$$

while

$$y_n(\alpha_i) = \frac{2\cos\left[\frac{(2n+1)}{2}\left(\frac{2i-1}{2n+1}\right)\pi\right] - \cos\left[\frac{(2n-1)}{2}\left(\frac{2i-1}{2n+1}\right)\pi\right]}{\cos\left(\frac{2i-1}{2n+1}\right)\pi/2} \quad (i=1,\ldots,n)$$

$$= \frac{2\cos(2i-1)\pi/2 - \cos\left[\left(1-\frac{2}{2n+1}\right)(2i-1)\pi/2\right]}{\cos\left(\frac{2i-1}{2n+1}\right)\pi/2} \quad (i=1,\ldots,n)$$

$$= \frac{-\cos\left[\left(1-\frac{2}{2n+1}\right)(2i-1)\pi/2\right]}{\cos\left(\frac{2i-1}{2n+1}\right)\pi/2} \quad (i=1,\ldots,n)$$

$$= \frac{-\cos\left[(2i-1)\pi/2\right]\cos\left[\frac{2i-1}{2n+1}\pi\right] - \sin\left[(2i-1)\pi/2\right]\sin\left[\frac{2i-1}{2n+1}\pi\right]}{\cos\left(\frac{2i-1}{2n+1}\right)\pi/2} \quad (i=1,\ldots,n)$$

$$= \frac{(-1)^i \sin\left[\left(\frac{2i-1}{2n+1}\right)\pi\right]}{\cos\left[\left(\frac{2i-1}{2n+1}\right)\pi/2\right]} \quad (i=1,\ldots,n) \; ,$$

which implies that

$$y_n(\alpha_i) < 0 \quad (i \text{ any odd integer between } 1 \text{ and } n)$$

and that

$$y_n(\alpha_i) > 0 \quad (i \text{ any even integer between } 1 \text{ and } n).$$

Since $y_n(\alpha)$ changes sign between 0 and α_1, α_1 and $\alpha_2,\ldots,\alpha_{n-1}$ and α_n, and since $y_n(\alpha)$ has only n zeros on the interval $(0,\pi)$, exactly one of the n zeros $\alpha_1',\ldots,\alpha_n'$ lies in each of the n intervals $(0,\alpha_1),\ldots,(\alpha_{n-1},\alpha_n)$. Hence, if the n zeros are so labeled that $\alpha_1' \leq \ldots \leq \alpha_n'$, it follows that

$$\alpha_i' \leq \alpha_i \qquad (i=1,\ldots,n) \; ,$$

whence

$$2\sqrt{\frac{k}{m}}\,\sin\left(\frac{\alpha_i'}{2}\right) = w_i' \leq w_i = 2\sqrt{\frac{k}{m}}\,\sin\left(\frac{\alpha_i}{2}\right) . \quad (i=1,\ldots,n)$$

The Effect of a Constraint on the Natural Frequencies

Let S and S' denote the systems in Figures 3(a) and 3(b), respectively, and let w_i and w'_i be the corresponding natural frequencies, where

$$w_1 \leq \dots \leq w_{n+1}$$

and

$$w'_1 \leq \dots \leq w'_n .$$

Note that system S' is the same as system S with the restriction $x_1 - x_2 = 0$ imposed on S. Theorem 5 then implies that

$$w_i \leq w'_i \leq w_{i+1} \qquad (i=1,\dots,n) .$$

This result will now be verified.

The natural frequencies of S are

$$w_i = 2\sqrt{\frac{k}{m}} \sin\left[\left(\frac{2i-1}{2n+3}\right)\frac{\pi}{2}\right] \qquad (i=1,\dots,n+1).$$

From the previous example the natural frequencies of S' are

$$w'_i = 2\sqrt{\frac{k}{m}} \sin\left(\frac{\alpha'_i}{2}\right) \qquad (i=1,\dots,n)$$

where $\alpha'_1,\dots,\alpha'_n$ $(0 < \alpha'_1 < \dots < \alpha'_n < \pi)$ are solutions to the equation

$$\frac{2\cos(n\alpha + \alpha/2) - \cos(n\alpha - \alpha/2)}{\cos\alpha/2} = 0 .$$

By reasoning not essentially different from that of the previous example, it follows that if

$$\alpha_i = \frac{2i-1}{2n+3}\pi \qquad (i=1,\dots,n+1),$$

then

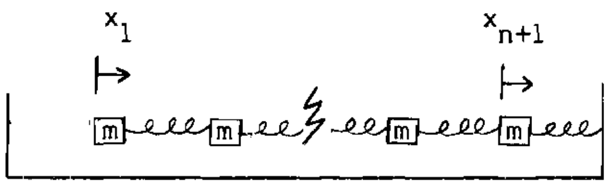

Figure 3(a)

The Original System S

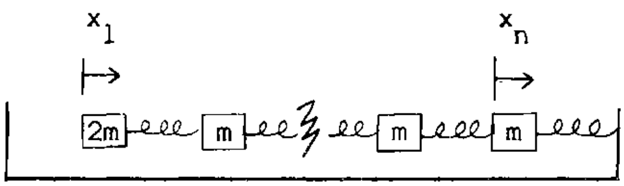

Figure 3(b)

The Constrained System S'

Figure 3. An Example Illustrating the Effect of a Constraint.

$$\alpha_i \le \alpha_i' \le \alpha_{i+1} \quad (i=1,\dots,n).$$

For

$$y_n(\alpha_i) = \frac{2\cos\left[\left(\frac{2n+1}{2}\right)\left(\frac{2i-1}{2n+3}\right)\pi\right] - \cos\left[\left(\frac{2n-1}{2}\right)\left(\frac{2i-1}{2n+3}\right)\pi\right]}{\cos\left(\frac{2i-1}{2n+3}\right)\pi/2} \quad (i=1,\dots,n+1)$$

$$= \frac{2\cos\left[\left(1-\frac{2}{2n+3}\right)(2i-1)\pi/2\right] - \cos\left[\left(1-\frac{4}{2n+3}\right)(2i-1)\pi/2\right]}{\cos\left(\frac{2i-1}{2n+3}\right)\pi/2} \quad (i=1,\dots,n+1)$$

$$= \frac{2\cos(2i-1)\pi/2\,\cos\left(\frac{2i-1}{2n+3}\right)\pi + 2\sin(2i-1)\pi/2\,\sin\left(\frac{2i-1}{2n+3}\right)\pi}{\cos\left(\frac{2i-1}{2n+3}\right)\pi/2}$$

$$- \frac{\cos(2i-1)\pi/2\,\cos\left[2\left(\frac{2i-1}{2n+3}\right)\pi\right] - \sin(2i-1)\pi/2\,\sin 2\left(\frac{2i-1}{2n+3}\right)\pi}{\cos\left(\frac{2i-1}{2n+3}\right)\pi/2} \quad (i=1,\dots,n+1)$$

$$= \frac{(-1)^{i+1}\,2\sin\left(\frac{2i-1}{2n+3}\right)\pi\left[1-\cos\left(\frac{2i-1}{2n+3}\right)\pi\right]}{\cos\left(\frac{2i-1}{2n+3}\right)\pi/2}, \quad (i=1,\dots,n+1)$$

which implies that

$$y_n(\alpha_i) > 0 \quad (i \text{ an odd integer between } 1 \text{ and } n+1)$$

and

$$y_n(\alpha_i) < 0 \quad (i \text{ an even integer between } 1 \text{ and } n+1).$$

Hence

$$\alpha_i \le \alpha_i' \le \alpha_{i+1}, \quad (i=1,\dots,n)$$

which implies that

$$w_i \le w_i' \le w_{i+1} \qquad (i=1,\dots,n).$$

A Numerical Example Illustrating Theorems 5 and 6

Denote the systems in Figures 4(a), 4(b), and 4(c) by S, S', and S'', respectively, and let w_i, w_i', and w_i'' be the corresponding natural frequencies. If the constraint $x_2 - x_3 = 0$ is applied to system S, then system S' is obtained; and if the constraint $x_1 = 0$ is applied to system S, system S'' arises. Theorem 5 then implies that

$$w_1 \leq w_1' \leq w_2 \leq w_2' \leq w_3 \leq w_3' \leq w_4$$

and

$$w_1 \leq w_1'' \leq w_2 \leq w_2'' \leq w_3 \leq w_3'' \leq w_4 .$$

Now observe that the potential energy is the same for systems S' and S'' for any common set of values of their coordinates but that the kinetic energy of S' is greater than or equal to the kinetic energy of S'' for any common set of velocities. Thus Theorem 6 implies that

$$w_i' \leq w_i'' \qquad (i=1,2,3,) .$$

This result combined with the previous result gives

$$w_1 \leq w_1' \leq w_1'' \leq w_2 \leq w_2' \leq w_2'' \leq w_3 \leq w_3' \leq w_3'' \leq w_4 .$$

Direct computation of the natural frequencies yields

$$w_1 = \sqrt{\frac{k}{m}} \cdot \sqrt{\frac{3-\sqrt{5}}{2}}, \quad w_1' = \sqrt{\frac{k}{m}} \cdot \sqrt{\frac{3-\sqrt{5}}{2}}, \quad w_1'' = \sqrt{\frac{k}{m}} \cdot \sqrt{2-\sqrt{2}}$$

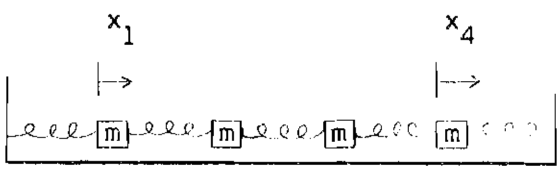

Figure 4(a)
The original system S

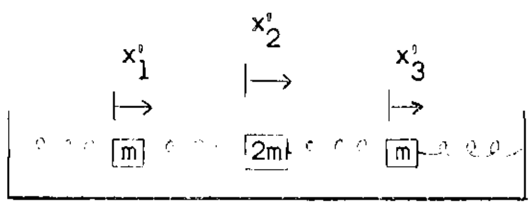

Figure 4(b)
The system S' derived from S
by applying the constraint $x_2 - x_3 = 0$

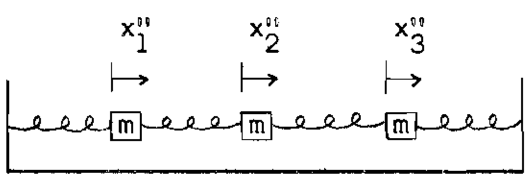

Figure 4(c)
The system S'' derived from S
by applying the constraint $x_1 = 0$

Figure 4. An Example Illustrating Theorems 5 and 6.

$$w_2 = \sqrt{\frac{k}{m}} \cdot \sqrt{2 - \sqrt{\frac{3 - \sqrt{5}}{2}}}\ , \quad w_2' = \sqrt{\frac{k}{m}} \cdot \sqrt{2}\ , \quad w_2'' = \sqrt{\frac{k}{m}} \cdot \sqrt{2}$$

$$w_3 = \sqrt{\frac{k}{m}} \cdot \sqrt{\frac{3 + \sqrt{5}}{2}}\ , \quad w_3' = \sqrt{\frac{k}{m}} \cdot \sqrt{\frac{3 + \sqrt{5}}{2}}\ , \quad w_3'' = \sqrt{\frac{k}{m}} \cdot \sqrt{2 + \sqrt{2}}$$

$$w_4 = \sqrt{\frac{k}{m}} \cdot \sqrt{2 + \sqrt{\frac{3 + \sqrt{5}}{2}}}$$

A comparison of these frequencies reveals that the numerical results stand in the relation predicted.

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